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Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **7472**

Unique Paper Code : **2352203602**

Name of the Paper : **DSC-Elementary Mathematical Analysis**

Name of the Course : **NEP-UGCF B.A. (Prog) with Mathematics as Major**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

All questions are compulsory.

1. (*a*) Define the limit of a function in terms of $\in$ and δ. Use $\in - \delta$ definition, to show that $\lim_{x \to -4} (2x + 13) = 5$. 7.5

(*b*) State sequential criterion for limits. Use it to prove that $\lim_{x_0 \in \mathbf{R}} f(x)$ does not exist, where : 7.5

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

P.T.O.

(c) Let $f : A \to \mathbf{R}$ where $A \subseteq \mathbf{R}$ and let x_0 be a cluster point of A. Then, prove that $\lim_{x \to x_0} f(x) = f(x_0)$ if and only if for each sequence $< x_n >$ in $A \setminus \{x_0\}$ that converges to x_0, the sequence $< f(x_n) >$ converges to $f(x_0)$. 7.5

2. (a) Define the continuity of a function at a point in terms of $\in - \delta$. Use $\in - \delta$ definition, to show that $f(x) = 2x^2 + 3x + 5$ is continuous at $x_0 = -1$. 7.5

(b) State Intermediate Value theorem. Also, prove that $x\,2^x = 1$ for some x in $(0, 1)$. 7.5

(c) Define uniform continuity of a function f on interval I. Use it to prove that the function $f(x) = x^3$ is uniformly continuous on the interval $[0, 4]$. 7.5

3. (a) Let f be defined and bounded on a compact interval $[a, b]$ for each partition P of $[a, b]$. Define the upper and lower Darboux's sums for f over P. Also, show that for all partitions P of $[a, b]$, $\underline{S}(f, P) \leq \overline{S}(f, P)$. 7.5

(b) Define Riemann Integrability of a bounded function on $[a, b]$. Prove that a constant function $f(x) = c$ is integrable over $[a, b]$ and $\int_a^b f = c(b - a)$. 7.5

(c) Consider the characteristic function of a closed interval $f = \chi_{[1, 3]}$ given by :

$$f(x) = \begin{cases} 1, & \text{if } 1 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Prove that f is integrable on $[0, 5]$ and find $\int_0^5 f$. 7.5

4. (*a*) Use sequential criterion for integrability to show that function $f(x) = x$ is integrable on $[0, 1]$ and also find $\int_0^1 f$. 7.5

(*b*) Prove that :

(*i*) If f is integrable on $[a, b]$ and if $x \in [a, b]$, $m \le f(x) \le \mathrm{M}$, then $m(b-a) \le \int_a^b f \le \mathrm{M}(b-a)$.

(*ii*) If f is integrable on $[a, b]$ and $\forall x \in [a, b]$, $|f(x)| \le \mathrm{M}$, then $\left|\int_a^b f\right| \le \mathrm{M}(b-a)$. 7.5

(*c*) Let $f(x) = 3x + 2$ defined on the interval $[1, 3]$ and the partition $\mathrm{P} = \left\{1, \frac{3}{2}, 2, 3\right\}$. Find $\underline{\mathrm{S}}(f, \mathrm{P})$ and $\overline{\mathrm{S}}(f, \mathrm{P})$. Also, find $\underline{\mathrm{S}}(f, \mathrm{P}) - \overline{\mathrm{S}}(f, \mathrm{P})$. 7.5

5. (*a*) Define pointwise convergence and find the pointwise limit of the following sequence of function :

(*i*) $< f_n >$, where $f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \le \frac{1}{n}: \\ |x| & \text{if } \frac{1}{n} < |x| \le 1 \end{cases}$ on $[-1, 1]$

(*ii*) $< g_n >$, where $g_n(x) = \frac{nx}{1+n^2x^2}$ on $\mathbf{R}$. 7.5

(*b*) Let $f_n(x) = e^{-nx}$, $x \ge 0$. Then show that $< f_n >$ is uniformly convergent on $[a, \infty]$, where $a > 0$ but is not uniformly convergent on $[0, \infty]$. 7.5

P.T.O.

(c) Let $< f_n >$ be a sequence of continuous functions that converges pointwise to a function f on $[a, b]$. Then does it mean that f is continuous on $[a, b]$? Justify your answer and also state the condition on convergence that ensures f is continuous on $[a, b]$. 7.5

6. (a) State Weierstarass M-Test to test the uniform convergence of a series of functions. Hence check the uniform convergence of the following series :

(i) $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^3}$ on $\mathbf{R}$

(ii) $\sum_{n=0}^{\infty} \dfrac{1}{1+n^2x^2}$ on $[1, 2]$. 7.5

(b) Find the Radius of convergence and Interval of convergence for the following Power Series : 7.5

(i) $\sum_{n=0}^{\infty} \dfrac{n^2}{2^n} (x-3)^n$

(ii) $\sum_{n=0}^{\infty} \dfrac{n!}{3^n} x^n$.

(c) Write Power Series expansion of $\ln(1 + x)$ and use Abel's theorem to show that $\ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots\ldots$. 7.5