

[This question paper contains 7 printed pages]

Your Roll No. :

Sl. No. of Q. Paper : **8083** **I**

Unique Paper Code : 2353010014

Name of the Paper : Integral Transforms

Name of the Course : **B.Sc.(H) Mathematics
(DSE)**

Semester : VI

Time : 3 Hours **Maximum Marks : 90**

Instructions for Candidates :

- (i) Write your Roll No. on the top immediately on receipt of this question paper.
- (ii) Attempt **all** questions by selecting **two** Parts from each question.
- (iii) **All** questions carry equal marks.
- (iv) **All** Parts from each question carry **7.5** marks.
- (v) Parts of questions to be attempted together.
- (vi) Use of calculator is not allowed.

P.T.O.

1. (a) Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

- (b) Obtain the complex Fourier series expansion for the function $f(x) = e^x$ in $[-\pi, \pi]$. Use the Parseval relation to show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(1+k^2)} = \pi \coth \pi.$$

- (c) Using the Fourier integral theorem, show that if

$$f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0, & x < 0, x > \pi \end{cases}$$

then

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos ax + \cos(a(\pi - x))}{1 - a^2} da, -\infty < x < \infty.$$

Hence show that
$$\int_0^{\infty} \frac{\cos\left(\frac{a\pi}{2}\right)}{1 - a^2} da = \frac{\pi}{2}.$$

2. (a) Show that the Fourier series of the triangular function defined by

$$f(x) = \begin{cases} \frac{2x}{\pi}, & 0 \leq x \leq \pi/2 \\ \frac{2}{\pi}(\pi - x), & \pi/2 \leq x \leq \pi \end{cases}$$

is

$$f(x) = \frac{8}{\pi^2} \left\{ \frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \dots \right\}.$$

Further show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$

- (b) State and prove Riemann-Lebesgue Lemma.
 (c) Determine the Fourier series expansion of x^2 for $-\pi < x < \pi$, by performing the integration of the appropriate Fourier series.
3. (a) If $\bar{f}(s)$ and $\bar{g}(s)$ are the Laplace transform of $f(t)$ and $g(t)$ respectively, then prove that :

(i) $\mathcal{L}[H(t-a)f(t-a)] = e^{-as} \mathcal{L}\{f(t)\}$

(ii) $\mathcal{L}[H(t-a)g(t)] = e^{-as} \mathcal{L}\{g(t+a)\}.$

Also, find out the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 2 \\ t-2, & t \geq 2 \end{cases}$$

- (b) Find the inverse Laplace transform of the following functions :

(i) $\frac{s}{(s^2 + a^2)(s^2 + b^2)}$

(ii) $\frac{1}{s(s+a)}$

- (c) Define convolution of two functions in Laplace transform and then prove the following :

(i) $f * g = g * f$

(ii) $\frac{d}{dt}[(f * g)(t)] = (f' * g)(t) + f(0)g(t)$

4. (a) Define the Fourier transform of a function $f(t)$ and show that :

$$f\{\exp(-a|x|)\} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + k^2} \right), a > 0$$

- (b) Define the Fourier cosine transform of a function $f(x)$. Let $f(x)$ and its first derivative vanish as $x \rightarrow \infty$. If $F_c(k)$ is the Fourier transform, then prove that:

$$f_c[f''(x)] = -k^2 f_c(k) - \sqrt{\frac{2}{\pi}} f'(0)$$

- (c) Prove the following :

$$(i) \int_{-\infty}^{\infty} (f * g)(x) dx = \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} g(v) dv$$

$$(ii) \int_{-\infty}^{\infty} F(k) g(k) dk = \int_{-\infty}^{\infty} f(x) G(x) dx,$$

where $F(k)$ and $G(x)$ are Fourier transforms of $f(x)$ and $g(k)$ respectively.

5. (a) Find the solution of the ordinary differential equation

$$-\frac{d^2 u}{dx^2} + a^2 u = f(x), \quad -\infty < x < \infty$$

by the Fourier transform method.

- (b) Let a string of length π be fixed at both the ends. The motion of the string due to a force acting on it is governed by

$$\begin{aligned}
 u_{tt} &= c^2 u_{xx} + f(x, t), & 0 < x < \pi, & \quad t > 0, \\
 u(x, 0) &= 0, & u_t(x, 0) &= 0, & \quad 0 < x < \pi, \\
 u(0, t) &= 0 & u(\pi, t) &= 0, & \quad t > 0
 \end{aligned}$$

Obtain the solution to the above equation of motion by applying the finite Fourier sine transform.

- (c) Obtain the solution of the following Cauchy Problem for the Diffusion Equation using Fourier transform.

$$u_t = k u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

where k is a diffusivity constant with the initial condition

$$u(x, 0) = f(x), \quad -\infty < x < \infty$$

6. (a) Use Laplace transform to find the solution of the following equation

$$x u_t + u_x = x, \quad x > 0, \quad t > 0$$

with the initial and boundary conditions

$$\begin{aligned}
 u(x, 0) &= 0 & \text{for } x > 0, \\
 u(0, t) &= 0 & \text{for } t > 0
 \end{aligned}$$

- (b) Use Laplace transform to find the solution of the system of equations

$$\frac{dx_1}{dt} - x_2 = 0,$$

$$\frac{dx_2}{dx} + 2x_1 - 3x_2 = 0$$

with $x_1(0) = 0$ and $x_2(0) = 1$.

- (c) Use Laplace transform to solve the equation

$$u_t = ku_{xx}, \quad x > 0, t > 0$$

with the initial and boundary conditions

$$u(x, 0) = 0, \quad x > 0$$

$$u(0, t) = f(t), \quad t > 0$$

$$u(x, t) \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad t > 0$$
