

(d) State D'Alembert's Ratio test for a series. Find if the series,

$$\frac{1}{2} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \dots \text{ is convergent.}$$

[This question paper contains 8 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 1382

I

Unique Paper Code : 2352011102

Name of the Paper : Elementary Real Analysis

Name of the Course : B.Sc. (H) Mathematics
(NEP-UGCF 2022)

Semester : I – DSC-2

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. Part of the questions to be attempted together.
4. **All** questions carry equal marks.
5. Use of Calculator is not allowed.

1. (a) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.
- (b) State the order properties of $\mathbb{R}$. Using it prove that if a, b, c are real numbers such that $a > b$, then $a + c > b + c$.
- (c) Find all values of x satisfying $|x - 2| \leq x + 1$.
- (d) Write the definition of Supremum and Infimum of a set. Give an example of a set having supremum and infimum, where the set
- (i) contains its supremum and infimum
 - (ii) does not contain its supremum and infimum
2. (a) State and prove Archimedean property.
- (b) Let S be a non-empty subset of $\mathbb{R}$ and $a > 0$, then show that

$$\sup(aS) = a \sup S$$

6. (a) State the Alternating Series test. Show that the

alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is convergent.

- (b) Test the convergence of the series

$$\frac{1}{e} + \frac{4}{e^2} + \frac{27}{e^3} + \frac{256}{e^4} + \frac{3,125}{e^5} + \dots$$

- (c) Define a conditionally convergent series and an absolutely convergent series. Test the series

$\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^{3/2}}$ for absolute or conditional convergence.

5. (a) State and prove Cauchy Criterion for convergence

of a series $\sum_{n=1}^{\infty} a_n$.

(b) Test the convergence of the following series :

(i) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

(c) Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, $p > 0$ is convergent for

$p > 1$ and divergent for $p \leq 1$.

(d) Show that if the series $\sum u_n$ converges, then

$\lim_{n \rightarrow \infty} u_n = 0$. Is the converse true? Justify your

answer.

(c) Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If

(a_n) is a sequence of positive real numbers with

$\lim_{n \rightarrow \infty} (a_n) = 0$ and for some constant $K > 0$ and some

$m \in \mathbb{N}$ we have $|x_n - x| \leq Ka_n$ for all $n \geq m$, then

prove that $\lim_{n \rightarrow \infty} (x_n) = x$.

(d) Using the definition of limit, show that

$$\lim_{n \rightarrow \infty} \left(\frac{4n+5}{3n+4} \right) = \frac{4}{3}.$$

3. (a) Let (x_n) and (y_n) be sequences of real number

such that $\lim_{n \rightarrow \infty} (x_n) = x$ and $\lim_{n \rightarrow \infty} (y_n) = y$, then show

that $\lim_{n \rightarrow \infty} (x_n + y_n) = x + y$.

(b) Let (x_n) be a sequence of positive real numbers

such that $L = \lim_{n \rightarrow \infty} \left(\frac{x_{n+1}}{x_n} \right)$ exists. Show that if $L < 1$,

then (x_n) converges and $\lim_{n \rightarrow \infty} (x_n) = 0$.

(c) State Squeeze theorem and show that if

$$z_n = (2^n + 3^n)^{\frac{1}{n}} \text{ then } \lim_{n \rightarrow \infty} z_n = 3.$$

(d) Let $X = (x_n)$ be a sequence of real numbers defined by $x_1 = 1$ and

$$x_{n+1} = \sqrt{2 + x_n} \text{ for } n \in \mathbb{R}.$$

Show that the sequence (x_n) is convergent and find its limit.

4. (a) Prove that if a sequence (x_n) is a monotone decreasing and bounded below sequence of real numbers, then it is convergent.

(b) State Bolzano Weierstrass Theorem for Sequences. Show that the sequence $((-1)^n)$ is divergent.

(c) Find limit inferior and limit superior of the following sequences:

$$(i) \left(\sin \left(\frac{n\pi}{4} \right) \right)$$

$$(ii) (3 + (-1)^n)$$

(d) Show that every Cauchy sequence of real numbers is bounded. Is the converse true? Justify your answer.