(c) Show by integrating the series for $\frac{1}{1+x}$ that $|x| < 1$, then $\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

(2000)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 3378 **I**

Unique Paper Code : 2354000008

Name of the Paper : Elementary Mathematical Analysis

Name of the Course : **COMMON PROG GROUP (GE)**

Semester : V

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.
2. Attempt all question by selecting two parts from each question.
3. Each question carries 15 marks.
4. Use of Calculator not allowed.

1. (a) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A and $f: A \to \mathbb{R}$ be a function, then define limit of function f at c. Use $\epsilon - \delta$ definition to show that $\lim_{x \to -4} (2x + 13) = 5$.

(b) State Sequential Criterion for limits. Use it to prove that 1 $\lim_{x \to 0} sin\left(\frac{1}{x}\right)$ does not exist.

(c) Prove that the Dirichlet function $f(x) = \begin{Bmatrix} 1, & if\ x\ is\ rational \\ 0, & if\ x\ is\ irrational \end{Bmatrix}$ is discontinuous everywhere.

2. (a) If $f: I \to \mathbb{R}$ is continuous on an interval I and $f(a).f(b) < 0$ for some $a, b \in I$. Prove that $f(x) = 0$ for some x between a and b.

(i) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ $on\ \mathbb{R}$

(ii) $\sum_{n=1}^{\infty} x^{-n}$ $on\ [2, \infty)$

(b) Find radius of convergence and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}$$

(ii) $f_n(x)=\frac{n}{x+n}$ on $[0, a]$ where $a > 0$

(b) Show that the sequence of functions $< f_n >$ is not uniformly convergent on $[0, \infty)$, where

$$f_n(x) = \frac{x}{x+n}; x \geq 0$$

(c) Give an example of a sequence of continuous functions that converges pointwise to a discontinuous function. What conditions on convergence ensures that the continuity of the limit function is preserved.

6. (a) State Weierstrass M-Test for uniform convergence of a series of functions. Use it to check the uniform convergence of:

(b) Define uniform continuity of a function f on an interval I. Prove that the function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

(c) Prove that for all partitions P of $\underline{S}(f,P) \leq \overline{S}(f,P)$. If $f(x) = 8x + 3$ on the interval $[0,1]$ and partition $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} 1\right\}$, verify that $\underline{S}(f,P) \leq \overline{S}(f,P)$.

3. (a) If Q is a refinement of a partition P, then prove that

(i) $\underline{S}(f,Q) \geq \underline{S}(f,P)$ and

(ii) $\overline{S}(f,Q) \leq \overline{S}(f,P)$

(b) Prove that a constant function $f(x) = c$ is integrable over [a, b], and $\int_a^b f = c(b-a)$.

(c) Prove that the Dirichlet function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \left\{ \begin{array}{l} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{array} \right\}$ is not integrable on any closed interval [a, b], where a < b.

4. (a) Use Sequential Criterion for integrability to show that the function $f(x) = x^2$ is integrable on [0,1] and find $\int_0^1 f$.

(b) Prove the following:

(i) If f is integrable on [a, b] and $\forall x \in$ [a, b], $f(x) \geq 0$, then $\int_a^b f \geq 0$.

(ii) If f is integrable on [a, b] and $\forall x \in$ [a, b], $|f(x)| \leq M$, then $|\int_a^b f| \leq M(b-a)$.

(c) Find pointwise limit of the following sequence of functions:

(i) $f_n(x) = \frac{nx}{1+n^2x^2}; x \in \mathbb{R}$

(ii) $f_n(x) = xe^{-nx}; x \in \mathbb{R}, x \geq 0$

5. (a) Show that the following sequence of functions is uniformly convergent on the specified domains.

(i) $f_n(x) = \frac{x}{n}$ on $[0,1]$