5. (a) (i) Prove that the set $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$ is a subring of the ring of all 2×2 matrices over $\mathbb{Z}$.

(ii) Prove that the ring $\mathbb{Z}_p$ of integers modulo p where p is a prime, is an integral domain.

(b) Define characteristic of a ring. Let R be a ring with unity 1. Prove that if 1 has infinite order under addition then charR = 0. If 1 has order n under addition, then charR = n.

(c) Prove that intersection of any collection of subrings of a ring R is a subring of R. Does the result hold for the union of subrings? Justify.

6. (a) State the ideal test. Let $\mathbb{R}[x]$ denote the set of all polynomials with real coefficients. Let $A = \{p(x) \in \mathbb{R}[x] \mid p(0) = 0\}$. Prove that A is a principal ideal of $\mathbb{R}[x]$.

(b) Determine all the ring homomorphisms from $\mathbb{Z}_n$ to itself.

(c) State and prove the First Isomorphism Theorem for rings.

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4014 **H**

Unique Paper Code : 2352572401

Name of the Paper : Abstract Algebra

Name of the Course : **B.A./B.Sc. (Programme) with Mathematics as Non-Major/ Minor – DSC**

Semester : IV

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** questions by selecting **two** parts from each question.

3. Each part carries **7.5** marks.

4. Use of Calculator not allowed.

1. (a) Let n be a fixed positive integer greater than 1. If a mod n = a' and b mod n = b', then prove that $(a + b) \bmod n = (a' + b') \bmod n$ and $(ab) \bmod n = (a'b') \bmod n$.

   (b) Define a Group. Show that the set {5,15, 25, 35} is a group under multiplication modulo 40. What is the identity element of this group? Is there any relationship between this group and U(8)?

   (c) Show that $G = GL(2, \mathbb{R})$, group of 2×2 matrices with real entries and nonzero determinants is non abelian. Also, construct a Cayley table for U(12).

2. (a) Let G be a group and H a nonempty subset of G. Prove that if ab^{-1} is in H whenever a and b are in H, then His a subgroup of G. Also, find the order of 7 in U(15).

   (b) Let G be a group and let $a \in G$. Prove that a and a^{-1} have the same order. Illustrate the above result in the group $\mathbb{Z}_{10}$.

   (c) Let a be an element of order n in a group G and let k be a positive integer. Show that

$$\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle \text{ and } |a^k| = \frac{n}{\gcd(n,k)}.$$

3. (a) If $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$.

   Show that $\alpha\beta \neq \beta\alpha$. Also compute the value of and find order of $\beta^{-1}\alpha\beta$.

   (b) Construct a complete Cayley table for D_4, the group of symmetries of a square. Find the inverse of each of the element in D_4.

   (c) Let H = {(1), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)}. Find all the left cosets of H in A_4, the alternating group of degree 4.

4. (a) State Lagrange's theorem for finite groups. Show that a group of prime order is cyclic.

   (b) State the normal subgroup test. Prove that $SL(2, \mathbb{R})$, the group of 2×2 matrices with determinant 1 is a normal subgroup of $GL(2, \mathbb{R})$, the group of 2×2 matrices with non zero determinant.

   (c) Let $f: G \to G'$ be a group homomorphism. Prove that if H be a cyclic subgroup of G then f(H) is a cyclic subgroup of G'. Prove or disprove that the map $g: (\mathbb{R}, +) \to (\mathbb{R}, +)$ given by $g(x) = x^2$ is a homomorphism.