

(b) Prove that a finite Integral Domain is a field.
Hence, show that $\mathbb{Z}_p$ is a field, where p is a prime number.

(c) State and prove the subring test.

Let $R = \left\{ \begin{bmatrix} a & a-b \\ a-b & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z})$ where $M_2(\mathbb{Z})$ is the ring of 2×2 matrices over $\mathbb{Z}$. Prove or disprove that R is a subring of $M_2(\mathbb{Z})$.

6. (a) Define an ideal of a ring R . State the ideal test.
Hence, prove that $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

(b) Let $f: R \rightarrow S$ be a ring homomorphism. Prove that

(i) $f(A)$ is a subring of S where A is a subring of R .

(ii) $f^{-1}(B) = \{r \in R \mid f(r) \in B\}$ is an ideal of R where B is an ideal of S .

(c) Determine all the ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$.

(3000)

[This question paper contains 4 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 4902

H

Unique Paper Code : 2352572401

Name of the Paper : Abstract Algebra

Name of the Course : B.A./B.Sc. (Programme) with
Mathematics as Non-Major/
Minor – DSC

Semester : IV

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. Each part carries **7.5** marks.
4. Use of Calculator not allowed.

1. (a) State Division Algorithm. Determine $51 \bmod 13$, $342 \bmod 85$, $62 \bmod 15$, $(82 \cdot 73) \bmod 7$, $(51+68) \bmod 7$ and $(35 \cdot 24) \bmod 11$.

P.T.O.

(b) Define a Group. Show that $G = \{1, -1, i, -i\}$ forms a group under complex multiplication.

(c) Let G be a group with the property that for any x, y, z in the group, $xy = zx$ implies $y = z$. Prove that G is Abelian. Also, in $GL(2, \mathbb{Z}_{13})$, find

$$\det \begin{bmatrix} 7 & 4 \\ 1 & 5 \end{bmatrix}.$$

2. (a) Let G be an abelian group and H and K be subgroups of G . Show that $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G . Also, find the order of 7 in $\mathbb{Z}_{10}$ under addition modulo 10.

(b) Let a be an element in a group G of order 30. Find $\langle a^{26} \rangle$, $\langle a^{17} \rangle$, $\langle a^{18} \rangle$ and $|a^{26}|$, $|a^{17}|$ and $|a^{18}|$.

(c) Find the order of each element of $U(15)$.

3. (a) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$ and

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$$

Write α , β and $\alpha\beta$ as a product of disjoint cycles. Also find β^{-1} .

(b) Construct a complete Cayley table for D_4 , the group of symmetries of a square. Is D_4 Abelian? Justify.

(c) Find all the left cosets of $\{1, 15\}$ in $U(32)$.

4. (a) Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbb{R})$? Justify.

(b) State Lagrange's theorem for finite groups. Prove that in a finite group, the order of each element of the group divides the order of the group.

(c) Let G be a group of permutations. For each σ in G , define

$$\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Prove that the function sgn is a homomorphism from G to the multiplicative group $\{-1, 1\}$. What is the Kernel?

5. (a) (i) Describe all the subrings of the ring of integers.

(ii) Let a belong to a ring R . Let $S = \{x \in R \mid ax = 0\}$. Show that S is a subring of R .