(c) Let L: $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that :

L([1,0,0]) = [1,0,0], L([0,1,0]) = [4,0,1], L([0,0,1]) = [0,–1,1].

Find L([2,5,–1]) and give a formula for L([x,y,z]) for any [x, y, z] $\in \mathbb{R}^3$.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5657 **H**

Unique Paper Code : 2354001202

Name of the Paper : Introduction to Linear Algebra

Name of the Course : **GE II**

Student : SOL

Semester : II

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.

1. (a) Find the initial point of the vector [–4, –1, 6] when it's terminal point is (2, 3, –2). Also, find a point that is two-thirds of the distance from the initial point to the terminal point.

(b) Prove that if x, y, z are mutually orthogonal vectors in $\mathbb{R}^n$, then

$$\|x + y + z\|^2 = \|x\|^2 + \|y\|^2 + \|z\|^2.$$

Verify it for the vectors x = [1, 0, 1], y = [0, 1, 0] and z = [–1, 0, 1].

(c) Solve the following homogenous system of linear equations by using Gauss-Jordan row reduction method :

$$2x + y + 4z = 0$$

$$3x + 2y + 5z = 0$$

$$-y + z = 0.$$

(c) Let L: $\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation given by $L([x_1, x_2, x_3]) = [0, x_2]$. Find a basis for ker(L) and range(L). Also, verify that dim(ker(L)) + dim(range(L)) = dim($\mathbb{R}^3$).

6. (a) Let L: $P_3 \to P_2$ be given by $L(ax^3 + bx^2 + cx + d) = 4cx^2 + (a - b)x + d$.

(i) Is $p(x) = 3x^3 + 3x^2$ in ker(L)? Justify the answer.

(ii) Is $p(x) = 5x^2 + 3x - 6$ in range(L)? Justify the answer.

(b) Let L: $M_{2\times2} \to M_{2\times2}$ be a linear transformation given by :

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & b+c \\ b-c & a \end{bmatrix}.$$

Determine whether the linear transformation L is one-one or not, and onto or not.

5. (a) Determine whether the following functions are linear transformations or not :

(i) $L_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $L_1([x_1, x_2, x_3]) = [2x_1 + x_2, x_3, x_2]$.

(ii) $L_2 : P_2 \rightarrow P_1$ given by $L_2(ax^2 + bx + c) = 2ax + b$.

(b) Let $L : M_{2\times2} \rightarrow \mathbb{R}^3$ be a linear transformation given by :

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [a - c - 2d, 2a + b - d, -2c + d].$$

Find the matrix representation for L with respect to the standard basis for $M_{2\times2}$ and $\mathbb{R}^3$ respectively.

Use this matrix to calculate $L\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\right)$ by matrix multiplication.

2. (a) Find the rank of the following matrix :

$$\begin{bmatrix} 3 & 5 & 2 \\ 4 & 2 & 3 \\ -1 & 2 & 4 \end{bmatrix}.$$

(b) Express the vector $x \in \mathbb{R}^3$ as a linear combination of the other vectors (if possible)

$x = [-18,15,16]$, $a_1 = [-4,1,2]$, $a_2 = [2,1,0]$, $a_3 = [6,-3,-4]$.

(c) Solve the following system of linear equations by using Gaussian elimination method:

$$3x - 3y - 2z = 23$$

$$-6x + 4y + 3z = -40$$

$$-2x + y + z = -12.$$

Also, indicate whether the system is consistent or inconsistent.

3. (a) Use the diagonalization method to determine whether

$$A = \begin{bmatrix} -18 & 40 \\ -8 & 18 \end{bmatrix}$$

is diagonalizable. If so, specify the matrices D and P such that $D = P^{-1}AP$.

(b) Let $V = \mathbb{R}^+$ be the set of all positive real numbers. Define the addition and scalar multiplication in V by $v_1 \oplus v_2 = v_1.v_2$ and $a \odot v = v^a$ for $v, v_1, v_2 \in V$ and $a \in \mathbb{R}$. Prove that V is a vector space over $\mathbb{R}$.

(c) Consider the subset $S = \{[1, 3, -1], [2, 7, -3], [4, 8, -7]\} \subset \mathbb{R}^3$. Use the simplified Span method to find the span(S). Also, find the dimension of span(S).

4. (a) Use the Independence test method to show that the following subset

$$S = \{[1,2,-1,1], [2,1,0,1], [2,-2,1,0], [11,1,1,4]\} \subset \mathbb{R}^4$$

is linearly dependent. Also, find a vector in S that can be expressed a linear combination of the other vectors in S.

(b) Check the given subsets A and B are the subspaces of $M_{2\times2}$ and $\mathbb{R}^3$, respectively, or not.

(i) $A = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2} \,\middle|\, a, b, c, d \in \mathbb{R},\ a + b + c + d = 0\right\}.$

(ii) $B = \{\alpha\,[1,3,2] \in \mathbb{R}^3 \mid \alpha \in \mathbb{R}\}$.

(c) Define a basis for a vector space V. Show that the set $S = \{[1,2,-3], [3,1,4], [2,-1,6]\}$ is a basis for $\mathbb{R}^3$.